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SYMMETRIC SIMPLE GAMES

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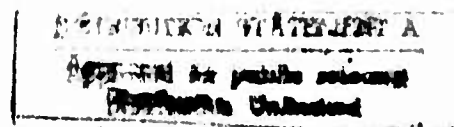
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13. ABSTRACT  A symmetric simple game is an $n$ -person game in which the winning coalitions are the coalitions of at least $k$ players. All von Neumann-Morgenstern stable sets, in which $(n-k)$ players are discriminated, are characterized for these games.			

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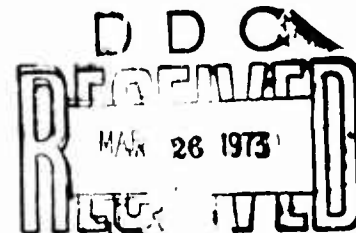
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A question of interest in the study of  $n$ -person games concerns conditions under which a set of players of a game may face discrimination which, in a sense, excludes them from the bargaining process. In [2, 4, 5], discriminatory solutions are given for several classes of games. In this paper, a characterization is given of all discriminatory solutions for  $n$ -person games in which all coalitions with at least  $k$  players win, and all coalitions with less than  $k$  players lose. The symmetric solutions of these games were given in [1] for the case  $k > \frac{n}{2}$ .

An  $n$ -person game is a function  $v$  from the coalitions (subsets) of a set of players  $N = \{1, 2, \dots, n\}$  to the reals satisfying

$$v(\emptyset) = v(\{i\}) = 0 \quad \text{for all } i \in N$$

$$0 \leq v(S) \leq 1 \quad \text{for all } S \subseteq N$$

$$v(N) = 1.$$

It is assumed throughout that  $n \geq 3$ . For any real vector  $x \in R^k$ , define  $x(S) = \sum_{i \in S} x_i$  and define  $x^S$  as the restriction of  $x$  to the coordinates in  $S$ . Let

$$X = \{x \in R^n: x(N) = 1, x_i \geq 0 \text{ for all } i \in N\}.$$

If  $x, y \in X$ , then  $y$  dominates  $x$  with respect to a non-empty coalition  $S$ , written  $y \text{ dom}_S x$ , if  $y^S > x^S$  and  $y(S) \leq v(S)$ . For  $A \subseteq X$ , define

$$\text{dom } A = \{x \in X: y \text{ dom}_S x \text{ for some } S \subseteq N, y \in A\}.$$

A solution, or von Neumann-Morgenstern stable set, is any set  $K \subseteq X$  satisfying

$$K \cup \text{dom } K = X, \tag{1}$$

$$K \cap \text{dom } K = \emptyset. \tag{2}$$

Any set  $K$  satisfying (1) is said to be externally stable; any set satisfying (2) is internally stable. Motivation for these definitions is given in [3].

A symmetric simple game, or  $(n,k)$ -game, is an  $n$ -person game satisfying

$$v(S) = 0 \quad \text{if } |S| < k$$

$$v(S) = 1 \quad \text{if } |S| \geq k,$$

where  $|S|$  denotes the number of players in the coalition  $S$ . Clearly domination in this game can occur only with respect to coalitions of at least  $k$  players. A  $p$ -discriminatory solution is a solution

$$D(\alpha_1, \dots, \alpha_p; i_1, \dots, i_p) = \{x \in X: x_{i_k} = \alpha_{i_k} \text{ for all } 1 \leq k \leq p\}.$$

The main result of this paper is a characterization of all  $m$ -discriminatory solutions for the  $(n, n-m)$ -game.

For  $m < \frac{n}{2}$ , let  $M \subseteq N$  be a set of  $m$  players, and let  $P = N - M$ . Also let  $\alpha$  be a non-negative  $m$ -vector, and write  $K(\alpha)$  for  $D(\alpha; M)$ .

Theorem.  $K(\alpha)$  is an  $m$ -discriminatory solution of the  $(n, n-m)$ -game if and only if

$$\alpha(M) + (n-m-1)\alpha_i < 1 \quad (3)$$

for all  $i \in M$ .

The proof follows a sequence of lemmas.

Lemma 1. For any  $\alpha$ ,  $K(\alpha)$  is internally stable.

Proof. Assume on the contrary that  $x, y \in K(\alpha)$  and  $y \text{ dom}_S x$ . Since  $y^M = \alpha = x^M$  and  $|S| \geq n-m$ , it follows that  $S = P$ . However,  $y(M) = x(M)$  and  $y(N) = x(N)$  imply  $y(P) = x(P)$  and therefore  $y_i \leq x_i$  for some  $i \in P$ , a contradiction.

Lemma 2. Suppose  $x \notin K(\alpha)$ . Then  $x \notin \text{dom } K(\alpha)$  if and only if

$$\alpha(M) + x(S \cap P) \geq 1 \quad \text{or} \quad x^{S \cap M} \neq \alpha^{S \cap M} \quad (4)$$

for all  $S \subseteq N$  with  $|S| = n-m$ .

Proof. If  $y \in K(\alpha)$  and  $y \text{ dom}_T x$ , then  $|T| \geq n-m$ . Take any  $S \subseteq T$  with  $|S| = n-m$ . Then  $y \text{ dom}_S x$ , and  $x^{S \cap M} < y^{S \cap M} = \alpha^{S \cap M}$ . Since  $2m < n$ ,  $S \cap P \neq \emptyset$  and therefore

$$\alpha(M) + x(S \cap P) < y(M) + y(S \cap P) < 1.$$

This establishes the sufficiency of (4). To establish necessity, assume that (4) fails for some  $S$ . Let

$$y_i = \alpha_i \quad i \in M$$

$$y_i = x_i + (1 - \alpha(M) - x(S \cap P)) / |S \cap P| \quad i \in S \cap P$$

$$y_i = 0 \quad i \in P - S.$$

Then  $y \in K(\alpha)$ ,  $y \text{ dom}_S x$  and therefore  $x \in \text{dom } K(\alpha)$ .

Lemma 3. Let  $\tau(x) = \{i \in M \mid x_i \geq \alpha_i\}$ . If  $\tau(x) = M$ , then  $x \in K(\alpha) \cup \text{dom } K(\alpha)$ . If  $\tau(x) \subsetneq M$ , then  $x \notin K(\alpha) \cup \text{dom } K(\alpha)$  if and only if

$$\alpha(M) + x(S \cap P) \geq 1 \quad (5)$$

for all  $S \subseteq N$  with  $|S| = n-m$  and  $\tau(x) \cap S = \emptyset$ .

Proof. Assume  $\tau(x) = M$ , and let  $\epsilon = x(M) - \alpha(M)$ . If  $\epsilon = 0$  then  $x \in K(\alpha)$ .

If  $\epsilon > 0$ , let



$$y_i = \alpha_i \quad i \in M$$

$$y_i = x_i + \epsilon/(n-m) \quad i \in P.$$

Then  $y \in K(\alpha)$  and  $y \text{ dom}_P x$ . The remainder of the lemma is simply a restatement of Lemma 2.

Lemma 4. There exists  $x \notin K(\alpha) \cup \text{dom } K(\alpha)$  with  $\tau(x) = T$  if and only if

$$\alpha(M) + |S \cap P| \cdot (1 - \alpha(T))/(n-m) \geq 1 \quad (6)$$

for all  $S \subseteq N$  such that  $|S| = n-m$  and  $S \cap T = \emptyset$ .

Proof. For any  $x \in X$ , let

$$y_i = \alpha_i \quad i \in T$$

$$y_i = 0 \quad i \in M-T$$

$$y_i = x_i + (x(M) - \alpha(T))/|P| \quad i \in P.$$

If  $x$  satisfies (5), then  $y$  clearly also satisfies (5). Therefore by Lemma 3, there exists  $x \notin K(\alpha) \cup \text{dom } K(\alpha)$  with  $\tau(x) = T$  if and only if there exists some  $y$  such that

$$y(P) = 1 - \alpha(T)$$

$$\alpha(M) + y(S \cap P) \geq 1 \quad (7)$$

for all  $S \subseteq N$  with  $|S| = n-m$  and  $S \cap T = \emptyset$ . By the symmetry of (7), such a  $y$  exists if and only if (7) is satisfied when

$$y_i = (1 - \alpha(T)) / |P| \quad i \in P.$$

This establishes the lemma.

Proof of theorem. Observe that (6) is satisfied if and only if it is satisfied when  $|S \cap P|$  is minimized, that is  $|S \cap P| = n - 2m + |T|$ . Therefore, in view of the preceding lemmas,  $K(\alpha)$  is externally stable if and only if

$$\alpha(M) + (n - 2m + t)(1 - \alpha(T)) / (n - m) < 1$$

for all  $T \subseteq M$ , where  $|T| = t$ . Replacing  $T$  with  $M - T$ , this condition is equivalent to

$$t \cdot \alpha(M) + (n - m - t)\alpha(T) < t \tag{8}$$

for all  $T \subseteq M$  with  $|T| = t > 0$ . For  $t = 1$ , this is exactly the condition (3) of the theorem. For  $t > 1$ , (3) implies

$$t \cdot \alpha(M) + (n - m - t)\alpha(T) < t \cdot \alpha(M) + t \cdot (n - m - 1)\bar{\alpha} < t,$$

where  $\bar{\alpha} = \max_{i \in M} (\alpha_i)$ . Thus (8) is equivalent to (3), completing the proof of the theorem.

#### Comments.

1. With slight modifications to the proof, the theorem may be shown to hold for all  $0 \leq m \leq n - 2$ .
2. The theorem characterizes all  $m$ -discriminatory solutions to the  $(n, n - m)$ -game. It is easily verified that the game has no  $k$ -discriminatory solutions for  $k \neq m$ .

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